

Physics 128 Discussion (Week 6)

Problem 1 (Quantum Statistical Mechanics)

In standard cosmology, it is crucial to compute the number density and energy density of relativistic particle species (bosons and fermions). Using quantum statistical mechanics.

(a) The number density n of bosonic and fermionic particles in thermal equilibrium follows the Bose-Einstein or Fermi-Dirac distribution. Assuming zero chemical potential ($\mu = 0$), derive the number density n in the relativistic limit ($m \ll T$) using the general formula

$$n = g \int \frac{d^3p}{(2\pi)^3} f(E), \quad (1)$$

where the statistical distribution function is given by

$$f(E) = \frac{1}{e^{E/T} \mp 1}, \quad (2)$$

with the $+$ sign for fermions and the $-$ sign for bosons.

For bosons with $\mu = 0$ and $T \gg m$

$$\begin{aligned} n_B &= g_B \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{E/T} - 1} \\ &\rightarrow g_B \int d\Omega \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^2}{e^{p/T} - 1} \quad (E = |\mathbf{p}|: \text{relativistic limit}) \\ &= g_B \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p/T} - 1} \\ &= g_B \frac{T^3}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^x - 1} \quad \left(x = \frac{p}{T}, dp = T dx\right) \\ &= g_B \frac{\Gamma(3) \zeta(3)}{2\pi^2} T^3 \\ &= g_B \frac{\zeta(3)}{\pi^2} T^3. \end{aligned} \quad (3)$$

For fermions with $\mu = 0$ and $T \gg m$

$$\begin{aligned} n_F &= g_F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{e^{E/T} + 1} \\ &\rightarrow g_F \int d\Omega \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^2}{e^{p/T} + 1} \quad (E = |\mathbf{p}|: \text{relativistic limit}) \\ &= g_F \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p/T} + 1} \\ &= g_F \frac{T^3}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^x + 1} \quad \left(x = \frac{p}{T}, dp = T dx\right) \\ &= g_F \frac{\zeta(3)}{2\pi^2} \frac{3}{2} T^3 \\ &= \frac{3}{4} g_F \frac{\zeta(3)}{\pi^2} T^3. \end{aligned} \quad (4)$$

(b) The energy density ρ is given by

$$\rho = g \int \frac{d^3p}{(2\pi)^3} E f(E). \quad (5)$$

Derive the result in the relativistic limit ($m \ll T$).

For bosons with $\mu = 0$ and $T \gg m$

$$\begin{aligned} \rho_B &= g_B \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E}{e^{E/T} - 1} \\ &\rightarrow g_B \int d\Omega \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^3}{e^{p/T} - 1} \quad (E = |\mathbf{p}|: \text{relativistic limit}) \\ &= g_B \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^3}{e^{p/T} - 1} \\ &= g_B \frac{T^4}{2\pi^2} \int_0^\infty dx \frac{x^3}{e^x - 1} \quad \left(x = \frac{p}{T}, dp = T dx\right) \\ &= g_B \frac{T^4}{2\pi^2} \frac{\pi^4}{15} \\ &= g_B \frac{\pi^2}{30} T^4. \end{aligned} \quad (6)$$

For fermions with $\mu = 0$ and $T \gg m$

$$\begin{aligned} \rho_F &= g_F \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E}{e^{E/T} + 1} \\ &\rightarrow g_F \int d\Omega \int_0^\infty \frac{dp}{(2\pi)^3} \frac{p^3}{e^{p/T} + 1} \quad (E = |\mathbf{p}|: \text{relativistic limit}) \\ &= g_F \frac{1}{2\pi^2} \int_0^\infty dp \frac{p^3}{e^{p/T} + 1} \\ &= g_F \frac{T^4}{2\pi^2} \int_0^\infty dx \frac{x^3}{e^x + 1} \quad \left(x = \frac{p}{T}, dp = T dx\right) \\ &= g_F \frac{T^4}{2\pi^2} \frac{3}{2} \frac{\pi^4}{120} \\ &= \frac{7}{8} g_F \frac{\pi^2}{30} T^4. \end{aligned} \quad (7)$$

(c) Before electron-positron annihilation, the universe contained relativistic neutrinos (fermions and $g_\nu = 2$) and photons (bosons and $g_\gamma = 2$). Compute the ratio of their number densities and energy densities using the standard results for bosons and fermions.

Now we have

$$n(T) = \begin{cases} g_B \frac{\zeta(3)}{\pi^2} T^3 & (\text{bosons}) \\ \frac{3}{4} g_F \frac{\zeta(3)}{\pi^2} T^3 & (\text{fermions}) \end{cases}, \quad \rho(T) = \begin{cases} g_B \frac{\pi^2}{30} T^4 & (\text{bosons}) \\ \frac{7}{8} g_F \frac{\pi^2}{30} T^4 & (\text{fermions}) \end{cases}. \quad (8)$$

Then we get the average one particle energies of a boson and a fermion are given by

$$\begin{aligned} \langle \varepsilon_B(T) \rangle &= \frac{\rho_B(T)}{n_B(T)} = \frac{\pi^4}{30\zeta(3)} T \approx 2.70T, \\ \langle \varepsilon_F(T) \rangle &= \frac{\rho_F(T)}{n_F(T)} = \frac{7}{6} \frac{\pi^4}{30\zeta(3)} T \approx 3.15T. \end{aligned} \quad (9)$$

Problem 2 (Neutrino Decoupling and Hubble Rate)

In standard cosmology, the decoupling of a particle species occurs when its interaction rate Γ becomes comparable to the Hubble rate H . We mainly consider relativistic particle species ($p = \rho/3$)

(a) When $\text{MeV} \lesssim T < M_W, M_Z$, neutrinos scatter primarily via weak interactions, with an approximate interaction rate

$$\Gamma(T) \sim G_F^2 T^5, \quad (10)$$

where G_F is the Fermi constant. During the radiation-dominated epoch and $\text{MeV} \lesssim T < M_W, M_Z$, the Hubble rate $H(T)$ is approximately

$$H(T) \approx 10 \frac{T^2}{M_{\text{Pl}}}, \quad (11)$$

where M_{Pl} is the Planck mass. Find the decoupling temperature T_{dec} by solving $\Gamma(T) = H(T)$.

The decoupling temperature T_{dec} is determined by setting $\Gamma(T) = H(T)$

$$\begin{aligned} G_F^2 T_{\text{dec}}^5 &= 10 \frac{T_{\text{dec}}^2}{M_{\text{Pl}}} \\ \Rightarrow T_{\text{dec}}^3 &= \frac{10}{G_F^2 M_{\text{Pl}}}. \end{aligned} \quad (12)$$

Thus

$$T_{\text{dec}} = \left(\frac{10}{G_F^2 M_{\text{Pl}}} \right)^{1/3}. \quad (13)$$

(b) Using $G_F \approx 1.166 \times 10^{-5} \text{ GeV}^{-2}$ and $M_{\text{Pl}} \approx 1.22 \times 10^{19} \text{ GeV}$, estimate the decoupling temperature T_{dec} numerically.

$$\begin{aligned} T_{\text{dec}} &= \left(\frac{10}{G_F^2 M_{\text{Pl}}} \right)^{1/3} \\ &\approx \left(\frac{10}{(1.166 \times 10^{-5} \text{ GeV}^{-2})^2 (1.22 \times 10^{19} \text{ GeV})} \right)^{1/3} \\ &\approx 2 \text{ MeV}. \end{aligned} \quad (14)$$

(c) After electron-positron annihilation, the photon temperature increases due to entropy conservation, while neutrinos remain unaffected. Derive the temperature ratio

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3}. \quad (15)$$

Using the present CMB temperature $T_{\gamma,0} \approx 2.725 \text{ K}$, compute the current neutrino temperature $T_{\nu,0}$. For the relativistic particles, the entropy density is given by

$$\begin{aligned} s(T) &= \frac{\rho + p}{T} \\ &= \frac{4}{3} \frac{\rho}{T} \quad \left(p = \frac{1}{3} \rho \right) \\ &= \frac{4}{3} g_{*s}(T) \left(\frac{\pi^2}{30} T^3 \right), \end{aligned} \quad (16)$$

where $g_{*s}(T)$ is the total effective relativistic degrees of freedom for entropy, *i.e.* the sum of the internal degrees of freedom for the particle species such that they are relativistic at universe's temperature T

$$g_{*s}(T) = \sum_{b:\text{boson}} g_b \left(\frac{T_b}{T} \right)^3 + \frac{7}{8} \sum_{f:\text{fermion}} g_f \left(\frac{T_f}{T} \right)^3. \quad (17)$$

Here T_b, T_f are the temperatures of each specific boson and fermion species when temperature of the universe is T . When $T = T_b = T_f$, we have

$$g_{*s}(T) = \sum_{b:\text{boson}} g_b + \frac{7}{8} \sum_{f:\text{fermion}} g_f. \quad (18)$$

Before e^+e^- annihilation,

$$\begin{aligned} g_{*s}^{\text{before}} &= g_\gamma + \frac{7}{8} (g_{e^-} + g_{e^+}) \\ &= 2 + \frac{7}{8} (2 + 2) \\ &= \frac{11}{2}, \end{aligned} \quad (19)$$

and after e^+e^- annihilation,

$$g_{*s}^{\text{after}} = g_\gamma = 2. \quad (20)$$

Now entropy density per comoving volume is conserved

$$\begin{aligned} s(T) &= \frac{4}{3} g_{*s}(T) \left(\frac{\pi^2}{30} T^3 \right) \propto g_{*s}(T) T^3 \\ \Rightarrow g_{*s}^{\text{before}} T_{\text{before}}^3 &= g_{*s}^{\text{after}} T_{\text{after}}^3, \end{aligned} \quad (21)$$

then using $T_{\text{before}} = T_\nu$ and $T_{\text{after}} = T_\gamma$, we get

$$\begin{aligned} \frac{T_\nu}{T_\gamma} &= \left(\frac{g_{*s}^{\text{after}}}{g_{*s}^{\text{before}}} \right)^{1/3} \\ &= \left(\frac{2}{11} \right)^{1/3} \\ &\simeq 0.71. \end{aligned} \quad (22)$$

Therefore,

$$T_\nu^0 = 0.71 T_\gamma^0 \sim 1.95 \text{ K} \quad (23)$$